Quadratics (2)

1. **(a)** Determine the restrictions for λ such that the equation $x^2 - 6x - 1 + \lambda(2x + 1) = 0$ has real roots.

(b) Determine the restrictions for λ such that the equation $\lambda x^2 - 6x - 1 + \lambda(x + 1) = 0$ has real roots.

2. **(a)** If α and β are the roots of the equation $ax^2 + 2bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, show that $\alpha + \beta = -\frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(b) If the above equation has real roots and m, n are real numbers such that $m^2 > n > 0$, show that the equation $ax^2 + 2mbx + nc = 0$ also has real solutions.

3. If α and β are the roots for the equation $px^2 + qx + r = 0$, where p, q, r are real numbers and $p, q \neq 0$, show that $\alpha + \beta = -\frac{q}{p}$, $\alpha\beta = \frac{r}{p}$

If γ and δ are the roots of $qx^2 + rx + p = 0$, show that

(a) $(\alpha - \gamma)(\alpha - \delta) = \frac{q\alpha^2 + r\alpha + p}{q}$

(b) $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = \frac{p^3 + q^3 + r^3 - 3pqr}{pq^2}$

(c) Hence or otherwise, determine the relationship needed for the above two quadratic equations to have one **common** real root.

4. If $ax^2 + 2bx + c = 0$ $(a \neq 0)$ and $y = x + \frac{1}{x}$, prove that

$$acy^2 + 2b(c + a)y + (a - c)^2 + 4b^2 = 0$$

Hence, if α and β are the roots of the equation $ax^2 + 2bx + c = 0$, show that

$$\left(\alpha + \frac{1}{\alpha}\right)^2 + \left(\beta + \frac{1}{\beta}\right)^2 = \frac{4b^2(a^2 + c^2) - 2ac(a - c)^2}{a^2c^2}$$

5. If x_1 and x_2 are the roots of the equation $x^2 - (a + d)x + ad - bc = 0$.

Prove that the equation with roots ${x_1}^3$ and ${x_2}^3$ is

$$x^2 - (a^3 + d^3 + 3abc + 3bcd)x + (ad - bc)^3 = 0$$

6. If α and β are the roots for the equation $x^2 + bx + c = 0$ and γ and δ are the roots for the equation $x^2 + \lambda bx + \lambda^2 c = 0$ where b, c and λ are real numbers, show that the equations with roots $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is $x^2 - \lambda b^2 x + 2\lambda^2 c(b^2 - 2c) = 0$ and show that the roots of this new equation are real.

7. If α and β are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ form the equation where the roots are $\frac{1}{\alpha - 4\beta}$ and $\frac{1}{\beta - 4\alpha}$.

8. If one of the roots for the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ is the positive square root of the other, show that $b^3 = ac(3b - a - c)$

Hence, find the value(s) of y such that the roots of the quadratic equation

$$27x^2 + 6x - (y + 2) = 0$$

has a root that is the positive square root of the other.

9. If α is a root of the equation $x^5 - 1 = 0$, where $\alpha \neq 1$, form a quadratic equation with roots $\alpha^4 + \alpha$ and $\alpha^3 + \alpha^2$.